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The 3-way intersection problem for $S(2,4,v)$ designs

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Abstract

In this paper the 3-way intersection problem for $S(2,4,v)$ designs is investigated. Let $b_v = \frac{v(v-1)}{12}$ and $I_3[v] = \{0, 1, \ldots, b_v\} \setminus \{b_v - 7, b_v - 6, b_v - 5, b_v - 4, b_v - 3, b_v - 2, b_v - 1\}$. Let $J_3[v] = \{k|$ there exist three $S(2,4,v)$ designs with k same common blocks$\}$. We show that $J_3[v] \subseteq I_3[v]$ for any positive integer $v \equiv 1, 4 \pmod{12}$ and $J_3[v] = I_3[v]$, for $v \geq 49$ and $v = 13$. We find $J_3[16]$ completely. Also we determine some values of $J_3[v]$ for $v = 25, 28, 37$ and 40.

KEYWORDS: 3-way intersection; $S(2,4,v)$ design; GDD; trade

1 Introduction

A *Stiener system* $S(2,4,v)$ is a pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a v-element set and $\mathcal{B}$ is a family of 4-element subsets of $\mathcal{V}$ called *blocks*, such that each 2-element subsets of $\mathcal{V}$ is contained in exactly one block.

Two Steiner systems $S(2,4,v)$, $(\mathcal{V}, \mathcal{B})$ and $(\mathcal{V}, \mathcal{B}_1)$ are said to *intersect* in s blocks if $|\mathcal{B} \cap \mathcal{B}_1| = s$. The intersection problem for $S(2,4,v)$ designs can be extended in this way: determine the sets $\overline{J_\mu[v]}(J_\mu[v])$ of all integers s such that there exists a collection of μ (≥ 2) $S(2,4,v)$ designs mutually intersecting in s blocks (in the same set of s blocks). This generalization is called μ-way intersection problem. Clearly $\overline{J_2[v]} = J_2[v] = J[v]$ and $J_\mu[v] \subseteq \overline{J_\mu[v]} \subseteq J[v]$.

The intersection problem for $\mu = 2$ was considered by Colbourn, Hoffman, and Lindner in [8]. They determined the set $J[v](J_2[v])$ completely

for all values $v \equiv 1, 4 \pmod{12}$, with some possible exceptions for $v =$ 25, 28 and 37. Let $[a, b] = \{a, a+1, ..., b-1, b\}$, $b_v = \frac{v(v-1)}{12}$, and $I[v] = [0, b_v] \setminus ([b_v - 5, b_v - 1] \cup \{b_v - 7\})$. It is shown in [8]; that:

(1) $J[v] \subseteq I[v]$ for all $v \equiv 1, 4 \pmod{12}$.

(2) $J[v] = I[v]$ for all admissible $v \geq 40$.

(3) $J[13] = I[13]$ and $J[16] = I[16] \setminus \{7, 9, 10, 11, 14\}$.

(4) $I[25] \setminus \{31, 33, 34, 37, 39, 40, 41, 42, 44\} \subseteq J[25]$ and $\{42, 44\} \not\subseteq J[25]$.

(5) $I[28] \setminus \{44, 46, 49, 50, 52, 53, 54, 57\} \subseteq J[28]$.

(6) $I[37] \setminus (\{64, 66, 76, 82, 84, 85, 88\} \cup [90, 94] \cup [96, 101]) \subseteq J[37]$.

Also Chang, Feng, and Lo Faro investigate another type of intersection which is called triangle intersection (See [3]). Milici and Quattrocchi [15] determined $J_3[v]$ for STSs. Other results about the intersection problem can be found in [1, 4, 5, 6, 10, 7, 13]. In this paper we investigate the three way intersection problem for $S(2, 4, v)$ designs. We set $I_3[v] = [0, b_v] \setminus [b_v - 7, b_v - 1]$. As our main result, we prove the following theorem.

Theorem 1.1 (1) $J_3[v] \subseteq I_3[v]$ *for all* $v \equiv 1, 4 \pmod{12}$.

(2) $J_3[v] = I_3[v]$ *for all admissible* $v \geq 49$.

(3) $I_3[40] \setminus \{b_{40} - 15, b_{40} - 14\} \subseteq J_3[40]$.

(4) $J_3[13] = I_3[13]$ *and* $J_3[16] = I_3[16] \setminus \{7, 9, 10, 11, 12\}$..

(5) $[0, 11] \cup \{13, 15, 17, 20, 29, 50\} \cup [22, 24] \subseteq J_3[25]$ *and* $\{42\} \notin J_3[25]$.

(6) $[1, 24] \cup \{27, 28, 33, 37, 39, 63\} \subseteq J_3[28]$.

(7) $\{18, 19, 78, 79, 81, 87, 102, 103, 111\} \cup [21, 32] \cup [34, 36] \cup [38, 43] \cup [45, 48] \cup [52, 54] \cup [58, 63] \cup [67, 71] \subseteq J_3[37]$.

2 Necessary conditions

In this section we establish necessary conditions for $J_3[v]$. For this purpose, we use another concept that is relative to intersection problem: A (v, k, t) *trade* of volume s consists of two disjoint collections T_1 and T_2, each of s blocks, such that for every t-subset of blocks, the number of blocks containing these elements (t-subset) are the same in both T_1 and T_2. A (v, k, t) trade of volume s is *Steiner* when for every t-subset of blocks, the number of blocks containing these elements are at most one. A μ-way (v, k, t) trade $T = \{T_1, T_2, \ldots, T_\mu\}$, $\mu \geq 2$ is a set of pairwise disjoint (v, k, t) trade.

In every collection the union of blocks must cover the same set of elements. This set of elements is called the *foundation* of the trade. Its notation is found (T) and $r_x =$ no. of blocks in a collection which contain the element x.

By definition of the trade, if $b_v - s$ is in $J_3[v]$, then it is clear that there exists a 3-way Steiner $(v, 4, 2)$ trade of volume s. Consider three $S(2, 4, v)$ designs (systems) intersecting in $b_v - s$ same blocks (of size four). The remaining set of blocks (of size four) form disjoint partial quadruple systems, containing

precisely the same pairs, and each has s blocks. Rashidi and Soltankhah in [16] established that there do not exist a 3-way Steiner $(v, 4, 2)$ trade of volume s, for $s \in \{1, 2, 3, 4, 5, 6, 7\}$. So we have the following lemma:

Lemma 2.1 $J_3[v] \subseteq I_3[v]$.

3 Recursive constructions

In this section we give some recursive constructions for the 3-way intersection problem. The concept of GDDs plays an important role in these constructions. Our aim of common blocks is the same common blocks in the sequel.

Let K be a set of positive integers . A *group divisible design* K-GDD (as GDD for short) is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of a finite set $\mathcal{X}$ into subsets (called groups); (2) $\mathcal{A}$ is a set of subsets of $\mathcal{X}$ (called blocks), each of cardinality from K, such that a group and a block contain at most one common element; (3) every pair of elements from distinct groups occurs in exactly one block.

If $\mathcal{G}$ contains u_i groups of size g_i, for $1 \leq i \leq s$, then we denote by $g_1^{u_1} g_2^{u_2} \ldots g_s^{u_s}$ the *group type* (or type) of the GDD. If $K = \{k\}$, we write $\{k\}$-GDD as k-GDD. A K-GDD of type 1^v is commonly called a *pairwise balanced design*, denoted by $(v, K, 1)$-PBD. When $K = \{k\}$ a PBD is just a Steiner system $S(2, k, v)$.
The following construction is a variation of Willson's Fundamental Construction.

Theorem 3.1 *(Weighting construction). Let $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ be a GDD with groups $G_1,\ G_2, \ldots,\ G_s$. Suppose that there exists a function $w\ :\ X \to Z^+ \cup \{0\}$ (a weight function) so that for each block $A = \{x_1, \ldots, x_k\} \in \mathcal{A}$ there exist three K-GDDs of type $[w(x_1),\ \ldots\ w(x_k)]$ with b_A common blocks. Then there exist three K-GDDs of type $[\sum_{x \in G_1} w(x), \ldots, \sum_{x \in G_s} w(x)]$ which intersect in $\sum_{A \in \mathcal{A}} b_A$ blocks.*

proof. For every $x \in \mathcal{X}$, let $S(x)$ be a set of $w(x)$ "copies" of x. For any $\mathcal{Y} \subset \mathcal{X}$, let $S(\mathcal{Y}) = \bigcup_{y \in \mathcal{Y}} S(y)$. For every block $A \in \mathcal{A}$, there exist three K-GDDs: $(S(A),\ \{S(x)\ :\ x \in A\},\ B_A)$, $(S(A),\ \{S(x)\ :\ x \in A\},\ \dot{B}_A)$, $(S(A),\ \{S(x)\ :\ x \in A\},\ \ddot{B}_A)$, which intersect in b_A blocks. Then it is readily checked that there exist three, K-GDDs: $(S(\mathcal{X}),\ \{S(G)\ :\ G \in \mathcal{G}\},\ \cup_{A \in \mathcal{A}} B_A)$, $(S(\mathcal{X}),\ \{S(G)\ :\ G \in \mathcal{G}\},\ \cup_{A \in \mathcal{A}} \dot{B}_A)$, $(S(\mathcal{X}),\ \{S(G)\ :\ G \in \mathcal{G}\},\ \cup_{A \in \mathcal{A}} \ddot{B}_A)$, which intersect in $\sum_{A \in \mathcal{A}} b_A$ blocks. ∎

Theorem 3.2 *(Filling construction (i)). Suppose that there exist three 4-GDDs of type $g_1 g_2 \ldots g_s$ which intersect in b blocks. If there exist three*

$S(2,4,g_i+1)$ designs with b_i common blocks for $1 \leq i \leq s$, then there exist three $S(2,4,\sum_{i=1}^{s} g_i+1)$ designs with $b+\sum_{i=1}^{s} b_i$ common blocks.

proof. It is obvious. ∎

Theorem 3.3 *(Filling construction (ii)). Suppose that there exist three 4-GDDs of type $g_1 g_2 \ldots g_s$ which intersect in b blocks. If there exist three $S(2,4,g_i+4)$ designs containing b_i common blocks for $1 \leq i \leq s$. Also all designs containing a block y. Then there exist three $S(2,4,\sum_{i=1}^{s} g_i+4)$ designs with $b+\sum_{i=1}^{s} b_i-(s-1)$ common blocks.*

proof. Let $(\mathcal{X},\mathcal{G},\mathcal{A}_1)$, $(\mathcal{X},\mathcal{G},\mathcal{A}_2)$ and $(\mathcal{X},\mathcal{G},\mathcal{A}_3)$ be three 4-GDDs of type $g_1 g_2 \ldots g_s$ which intersect in b blocks. Let $\mathcal{Y}=\{y_1,y_2,y_3,y_4\}$ be a set of cardinality 4 such that $\mathcal{X}\cap\mathcal{Y}=\phi$.
For $1 \leq i \leq s$, there exist three $S(2,4,g_i+4)$ designs $(g_i \cup \mathcal{Y},\varepsilon_{1i})$, $(g_i \cup \mathcal{Y},\varepsilon_{2i})$ and $(g_i \cup \mathcal{Y},\varepsilon_{3i})$ containing the same block $y=y_1,y_2,y_3,y_4$ with b_i common blocks. It is easy to see that $(\mathcal{X}\cup\mathcal{Y},\mathcal{A}_1\cup(\bigcup_{1\leq i\leq s-1}(\varepsilon_{1i}-y))\cup\varepsilon_{1s})$, $(\mathcal{X}\cup\mathcal{Y},\mathcal{A}_2\cup(\bigcup_{1\leq i\leq s-1}(\varepsilon_{2i}-y))\cup\varepsilon_{2s})$ and $(\mathcal{X}\cup\mathcal{Y},\mathcal{A}_3\cup(\bigcup_{1\leq i\leq s-1}(\varepsilon_{3i}-y))\cup\varepsilon_{3s})$ are three $S(2,4,\sum_{i=1}^{s} g_i+4)$ designs with $b+\sum_{i=1}^{s} b_i-(s-1)$ common blocks. ∎
We apply another type of recursive constructions that explained in the following.
Let there be three $S(2,4,v)$ designs with a common parallel class, then $J_{p3}[v]$ for $v \equiv 4 \pmod{12}$ denotes the number of blocks shared by these $S(2,4,v)$ designs , in addition to those shared in the parallel class.

Lemma 3.4 *Let G be a GDD on $v=3s+6t$ elements with b blocks of size 4 and group type $3^s 6^t$, $s \geq 1$. For $1 \leq i \leq b$, let $a_i \in J_{p3}[16]$. For $1 \leq i \leq s-1$, let $c_i+1 \in J_3[16]$ and let $c_s \in J_3[16]$. For $1 \leq i \leq t$, let $d_i+1 \in J_3[28]$. Then there exist three $S(2,4,4v+4)$ designs with precisely $\sum_{i=1}^{b} a_i+\sum_{i=1}^{s} c_i+\sum_{i=1}^{t} d_i$ blocks in common.*

proof. The proof is similar to Lemma 3.3 in [8]. ∎
The *flower* of an element is the set of blocks containing that element. Let $J_{f3}[v]$ denote the number of blocks shared by three $S(2,4,v)$ designs, in addition to those in a required common flower.

Lemma 3.5 *Let G, B be a GDD of order v with b_4 blocks of size 4, b_5 blocks of size 5 and group type $4^s 5^t$. For $1 \leq i \leq b_4$, let $a_i \in J_{f3}[13]$. For $1 \leq i \leq b_5$, let $c_i \in J_{f3}[16]$. For $1 \leq i \leq s$, let $d_i \in J_3[13]$ and for $1 \leq i \leq t$, let $e_i \in J_3[16]$. Then there exist three $S(2,4,3v+1)$ designs intersecting in precisely $\sum_{i=1}^{b_4} a_i+\sum_{i=1}^{b_5} c_i+\sum_{i=1}^{s} d_i+\sum_{i=1}^{t} e_i$ blocks.*

proof. The proof is similar to Lemma 3.5 in [8]. ■

Lemma 3.6 *[9]. The necessary and sufficient conditions for the existence of a 4-GDD of type g^n are: (1) $n \geq 4$, (2) $(n-1)g \equiv 0 \pmod 3$, (3) $n(n-1)g^2 \equiv 0 \pmod{12}$, with the exception of $(g,n) \in \{(2,4),(6,4)\}$, in which case no such GDD exists.*

Lemma 3.7 *[2]. There exists a $(v, \{4, 7^*\}, 1)$-PBD with exactly one block of size 7 for any positive integer $v \equiv 7, 10 \pmod{12}$ and $v \neq 10, 19$.*

Lemma 3.8 *[9]. A 4-GDD of type $12^u m^1$ exists if and only if either $u = 3$ and $m = 12$, or $u \geq 4$ and $m \equiv 0 \pmod 3$ with $0 \leq m \leq 6(u-1)$.*

4 Ingredients

In this section we discuss some small cases needed for general constructions.

Lemma 4.1 $J_3[13] = I_3[13]$.

proof. Construct an $S(2,4,13)$ design, $(\mathcal{V}, \mathcal{B})$ with $\mathcal{V} = \mathcal{Z}_{10} \cup \{a,b,c\}$. All blocks of $\mathcal{B}$ are listed in the following, which can be found in Example 1.26 in [9].
0 0 0 0 1 1 1 2 2 3 3 4 5
1 2 4 6 2 5 7 3 6 4 7 8 9
3 8 5 a 4 6 b 5 7 6 8 9 a
9 c 7 b a 8 c b 9 c a b c
Consider the following permutations on $\mathcal{V}$.

π_1	π_2	π_3	int. no.
id	$(0,1,2,3,4,5)$	$(5,4,3,2,1,0)$	0
id	$(8,5)(a,b)(3,7)(1,6)$	$(8,6)(1,5)(a,3,b,7)$	1
id	$(7,b,c,6)$	$(8,5,6,7)$	2
id	$(4,7)(9,2)(1,8)$	$(3,9,c)(1,8)$	3
id	$(3,7)(c,0,2)(1,6)(9,b)$	$(9,4)(3,7)(0,2)(1,6)$	4
id	$(a,b)(4,5)$	$(a,b)(c,8)$	5
id	id	id	13

■

Lemma 4.2 $J_3[16] = I_3[16] \setminus \{7, 9, 10, 11, 12\}$.

proof. The proof has three steps:

(1) $J_3[16] \subseteq J[16] = \{0, 1, 2, 3, 4, 5, 6, 8, 12, 20\}$.

(2) Construct an $S(2, 4, 16)$ design, $(\mathcal{V}, \mathcal{B})$ with $\mathcal{V} = \mathcal{Z}_{10} \cup \{a, b, c, d, e, f\}$. All 20 blocks of $\mathcal{B}$ are listed in the following, which can be found in Example 1.31 in [9].

$\{0, 1, 2, 3\}$, $\{0, 4, 5, 6\}$, $\{0, 7, 8, 9\}$, $\{0, a, b, c\}$, $\{0, d, e, f\}$, $\{1, 4, 7, a\}$, $\{1, 5, b, d\}$, $\{1, 6, 8, e\}$, $\{1, 9, c, f\}$, $\{2, 4, c, e\}$, $\{2, 5, 7, f\}$, $\{2, 6, 9, b\}$, $\{2, 8, a, d\}$, $\{3, 4, 9, d\}$, $\{3, 5, 8, c\}$, $\{3, 6, a, f\}$, $\{3, 7, b, e\}$, $\{4, 8, b, f\}$, $\{5, 9, a, e\}$, $\{6, 7, c, d\}$.

Consider the following permutations on $\mathcal{V}$.

π_1	π_2	π_3	int. no.
id	$(0, 1, 4, 9, d)(6, 5)(b, f)$	$(2, 3, 7, 6, c)(5, 8)(a, b)(f, 4)$	0
id	$(0, 1, 2, 3)(8, 9, 5, b, c)$	$(d, f, e, a)(4, 6, 7)(c, 9)$	1
id	$(1, 2)(a, b)(7, f)(c, e)(6, 8)$	$(a, b, 7, f, c, e, 6, 8)(4, d)$	2
id	$(c, e)(5, 6)(b, f, 7, d, 9)$	$(8, b, 7, a, d, 9, f)(1, 3)$	3
id	$(0, 1, 2, 3)$	$(4, 8, b, f)$	4
id	$(4, f)(d, 7)(a, 9, 5)$	$(7, c, d)(f, b, 4)$	5
id	$(0, 1, 2)(a, e)$	$(f, b)(1, 0, 2)$	6
id	$(6, 7, c)$	$(6, c, 7)$	8
id	id	id	20

Hence we have $\{0, 1, 2, 3, 4, 5, 6, 8, 20\} \subset J_3[16]$.

(3) $b_{16} - 8 \notin J_3[16]$:

If $b_{16} - 8 = 20 - 8 = 12 \in J_3[16]$, then a 3-way $(v, 4, 2)$ trade of volume 8 is contained in the $S(2, 4, 16)$ design. Let T be this trade.

If all elements in found (T) appear 3 times in T_i , then for one block as $a_1a_2a_3a_4$, there exist 8 more blocks, so $|T_i| \geq 9$. Hence there exists $x \in$ found (T), with $r_x = 2$. Without loss of generality, let $xa_1a_2a_3$ and $xb_1b_2b_3$ be in T_1. But T is Steiner trade so there exist (for example): $xb_1a_2a_3$ and $xa_1b_2b_3$ in T_2 and there exist $xa_1b_3a_3$ and $xb_1b_2a_2$ in T_3. Now T_1 must contains at least 6 pairs: a_1b_2, a_1b_3, a_2b_1, a_3b_1, a_3b_3, a_2b_2 which those come in disjoint blocks, since T is Steiner. So we have:

T_1	T_2	T_3
$xa_1a_2a_3$	$xb_1a_2a_3$	$xa_1b_3a_3$
$xb_1b_2b_3$	$xa_1b_2b_3$	$xb_1b_2a_2$
a_1b_2	$--$	$--$
a_1b_3	$--$	$--$
a_2b_1	$--$	$--$
a_2b_2	$--$	$--$
a_3b_1	$--$	$--$
a_3b_3	$--$	$--$

We know that the $S(2,4,16)$ design is unique (See [9]). Without loss of generality, we can assume $x, a_1, a_2, a_3 = 0, 1, 2, 3$ and $x, b_1, b_2, b_3 = 0, 4, 5, 6$ (two blocks of the $S(2,4,16)$ design). Hence T has the following form:

T_1	T_2	T_3
0123	0423	0163
0456	0156	0452
$15bd$	$--$	$--$
$168e$	$--$	$--$
$24ce$	$--$	$--$
$257f$	$--$	$--$
$349d$	$--$	$--$
$36af$	$--$	$--$

Therefore $r_7 = 1$, and by Lemma 3 in [11] this is impossible . ■

Lemma 4.3 $\{0, 23, 29, 50\} \subseteq J_3[25]$.

proof. Construct an $S(2,4,25)$ design, $(\mathcal{V}, \mathcal{B})$ with $\mathcal{V} = \mathcal{Z}_{10} \cup \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o\}$. All 50 blocks of $\mathcal{B}$ are listed in the following, which can be found in Example 1.34 in [9].
$\{0,1,2,i\}$, $\{0,l,3,6\}$, $\{0,4,8,o\}$, $\{0,a,5,9\}$, $\{0,7,g,h\}$, $\{0,b,d,n\}$, $\{0,c,f,g\}$, $\{0,k,m,e\}$, $\{1,3,a,b\}$, $\{1,4,7,m\}$, $\{1,5,6,o\}$, $\{1,8,f,h\}$, $\{1,9,e,l\}$, $\{1,c,k,n\}$, $\{1,d,g,j\}$, $\{2,3,7,o\}$, $\{2,4,b,9\}$, $\{2,8,5,n\}$, $\{2,6,f,g\}$, $\{2,a,c,m\}$, $\{2,d,k,l\}$, $\{2,e,h,j\}$, $\{3,4,5,j\}$, $\{3,8,c,d\}$, $\{3,9,k,f\}$, $\{3,e,g,n\}$, $\{3,h,i,m\}$, $\{4,6,d,e\}$, $\{4,a,g,k\}$, $\{h,4,c,l\}$, $\{4,i,n,f\}$, $\{5,7,c,e\}$, $\{5,b,h,k\}$, $\{5,d,f,m\}$, $\{5,g,i,l\}$, $\{6,7,8,k\}$, $\{6,9,c,i\}$, $\{6,a,h,n\}$, $\{6,b,j,m\}$, $\{9,7,j,n\}$, $\{7,a,d,i\}$, $\{7,b,f,l\}$, $\{m,g,8,9\}$, $\{8,a,j,l\}$, $\{8,b,e,i\}$, $\{9,d,h,o\}$, $\{a,e,f,o\}$, $\{b,c,g,o\}$, $\{i,j,k,o\}$, $\{l,m,n,o\}$.
Consider the following permutations on $\mathcal{V}$.

π_1	π_2	π_3	int. no.
id	$(0,1,2,3)$	$(2,1,0,3)$	23
id	$(0,1,2)$	$(2,1,0)$	29
id	id	id	50

Hence we have $\{23, 29, 50\} \subset J_3[25]$.
By taking the 5th, 6th and 8th designs, of Table 1.34 in [9], we have $0 \in J_3[25]$. ■

Lemma 4.4 $\{1, 63\} \subseteq J_3[28]$.

proof. $63 \in J_3[28]$ by taking an $S(2,4,28)$ design thrice. We obtain $1 \in J_3[28]$, by applying the following permutations on the design of Lemma 6.3 (Step 1) in the last section.
π_1 is identity, $\pi_2 = (0,1,3,5,6,7,12,17,15,18,19,20,11)(25,26,27)$, and $\pi_3 = \pi_2^{-1}$. ■

Lemma 4.5 *There exist three 4-GDDs of type* 4^4 *with* i *common blocks,* $i \in \{0,1,2,4,16\}$.

proof. Take the $S(2,4,16)$ design, $(\mathcal{V}, \mathcal{B})$ constructed in Lemma 4.2. Consider the parallel class $\mathcal{P} = \{\{0,1,2,3\},\{4,8,b,f\},\{5,9,a,e\},\{6,7,c,d\}\}$ as the groups of GDD to obtain a 4-GDD of type 4^4 $(\mathcal{X}, \mathcal{G}, \mathcal{B}')$, where $\mathcal{X} = \mathcal{V}$, $\mathcal{G} = \mathcal{P}$ and $\mathcal{B}' = \mathcal{B} \setminus \mathcal{P}$.
Consider the following permutations on $\mathcal{X}$, which keep $\mathcal{G}$ invariant.

π_1	π_2	π_3	int. no.
id	id	id	16
id	$(6,7,c)$	$(6,c,7)$	4
id	$(0,1,2)(a,e)$	$(f,b)(1,0,2)$	2
id	$(4,f)(d,7)(a,9,5)$	$(7,c,d)(f,b,4)$	1
id	$(0,1,2,3)$	$(4,8,b,f)$	0

In fact $J_{p3}[16]$ is precisely the intersection sizes of three 4-GDDs of group type 4^4 having all groups in common. ■

Corollary 4.6 $\{0,1,2,4,16\} \subseteq J_{p3}[16]$.

Lemma 4.7 *There exist three 4-GDDs of type* 3^5 *with* i *common blocks,* $i \in \{0,1,3,15\}$.

proof. Take the $S(2,4,16)$ design, $(\mathcal{V},\mathcal{B})$ constructed in Lemma 4.2. Delete the element 0 from this design to obtain a 4-GDD of type 3^5 $(\mathcal{X},\mathcal{G},\mathcal{B}')$, where $\mathcal{X}=\mathcal{V}\setminus\{0\}$,
$\mathcal{G}=\{\{1,2,3\},\{4,5,6\},\{7,8,9\},\{a,b,c\},\{d,e,f\}\}$ and $\mathcal{B}'=\mathcal{B}\setminus\{B\in\mathcal{B}: 0\in B\}$. Consider the following permutations on $\mathcal{X}$, which keep $\mathcal{G}$ invariant.

π_1	π_2	π_3	int. no.
id	id	id	15
id	$(a,c)(1,3)(6,5)$	$(7,9)(d,e)(2,3)(a,c)$	1
id	$(2,3)(5,6)(7,8)(a,c)(d,f)$	$(1,2)(4,6)(8,9)(a,c)(d,e)$	0

If we delete d then we have a 4-GDD of type 3^5 $(\mathcal{X},\mathcal{G},\mathcal{B}')$, where $\mathcal{X}=\mathcal{V}\setminus\{d\}$,
$\mathcal{G}=\{\{6,7,c\},\{2,8,a\},\{1,5,b\},\{3,4,9\},\{0,e,f\}\}$ and $\mathcal{B}'=\mathcal{B}\setminus\{B\in\mathcal{B}: d\in B\}$. When the following permutations act on $\mathcal{X}$ then we obtain 3 as intersection number.
$\pi_1=$ identity, $\pi_2=(6,7,c)$, $\pi_3=(6,c,7)$.

We have $\{0,1,3,15\}\subseteq J_{f3}[16]$ because $J_{f3}[16]$ is precisely the intersection sizes of three 4-GDDs of group type 3^5 having all groups in common. ■

Lemma 4.8 *There exist three 4-GDDs of type* 3^4 *with* i *common blocks,* $i\in\{0,1,9\}$.

proof. Take the $S(2,4,13)$ design, $(\mathcal{V},\mathcal{B})$ constructed in Lemma 4.1. Delete the element 0 from the design to obtain a 4-GDD of type 3^4 $(\mathcal{X},\mathcal{G},\mathcal{B}')$, where $\mathcal{X}=\mathcal{V}\setminus\{0\}$,
$\mathcal{G}=\{\{1,3,9\},\{2,8,c\},\{4,5,7\},\{6,a,b\}\}$ and $\mathcal{B}'=\mathcal{B}\setminus\{B\in\mathcal{B}: 0\in B\}$. Consider the following permutations on $\mathcal{X}$, which keep $\mathcal{G}$ invariant.

π_1	π_2	π_3	int. no.
id	id	id	9
id	$(a,b)(4,5)$	$(a,b)(8,c)$	1

If we delete 8 then we have a 4-GDD of type 3^4 $(\mathcal{X},\mathcal{G},\mathcal{B}')$, where $\mathcal{X}=\mathcal{V}\setminus\{8\}$,
$\mathcal{G}=\{\{0,2,c\},\{1,5,6\},\{3,7,a\},\{b,4,9\}\}$ and $\mathcal{B}'=\mathcal{B}\setminus\{B\in\mathcal{B}: 8\in B\}$. When the following permutations act on $\mathcal{X}$ then we obtain 0 as intersection number.
$\pi_1=$ identity, $\pi_2=(3,7)(c,0,2)(1,6)(9,b)$, $\pi_3=(9,4)(3,7)(0,2)(1,6)$.
We obtain $\{0,1,9\}\subset J_{f3}[13]$ because $J_{f3}[13]$ is precisely the intersection

sizes of three 4-GDDs of group type 3^4 having all groups in common. ■

Corollary 4.9 $\{0, 1, 9\} \subseteq J_{f3}[13]$.

5 Applying the recursions

In this section, we prove the main theorem for all $v \geq 40$. First we treat the (easier) case $v \equiv 1 \pmod{12}$.

Theorem 5.1 *For any positive integer* $v = 12u + 1$, $u \equiv 0, 1 \pmod 4$ *and* $u \geq 4$, $J_3[v] = I_3[v]$.

proof. Start from a 4-GDD of type 3^u from Lemma 3.6. Give each element of the GDD weight 4. By Lemma 4.5 there exist three 4-GDDs of type 4^4 with α common blocks, $\alpha \in J_{p3}[16]$. Then apply construction 3.1 to obtain three 4-GDDs of type 12^u with $\sum_{i=1}^{b} \alpha_i$ common blocks, where $b = \frac{3u(u-1)}{4}$ and $\alpha_i \in J_{p3}[16]$, for $1 \leq i \leq b$. By construction 3.2, filling in the groups by three $S(2, 4, 13)$ designs with $\beta_j (1 \leq j \leq u)$ common blocks, we have three $S(2, 4, 12u + 1)$ designs with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u} \beta_j$ common blocks, where $\beta_j \in J_3[13]$ for $1 \leq j \leq u$. It is checked that for any integer $n \in I_3[v]$, n can be written as the form of $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u} \beta_j$, where $\alpha_i \in J_{p3}[16](1 \leq i \leq b)$ and $\beta_j \in J_3[13](1 \leq j \leq u)$. ■

Theorem 5.2 *For any positive integer* $v = 12u + 1$, $u \equiv 2, 3 \pmod 4$ *and* $u \geq 7$, $J_3[v] = I_3[v]$.

proof. There exists a $(3u + 1, \{4, 7^*\}, 1)$-PBD from Lemma 3.7, which contains exactly one block of size 7. Take an element from the block of size 7. Delete this element to obtain a 4-GDD of type $3^{u-2}6^1$. Give each element of the GDD weight 4. By Lemma 4.5, there exist three 4-GDDs of type 4^4 with α common blocks, $\alpha \in J_{p3}[16]$. Then apply construction 3.1 to obtain three 4-GDDs of type $12^{u-2}24$ with $\sum_{i=1}^{b} \alpha_i$ common blocks, where $b = \frac{3(u^2-u-2)}{4}$ and $\alpha_i \in J_{p3}[16]$ for $1 \leq i \leq b$. By construction 3.2, filling in the groups by three $S(2, 4, 13)$ designs with $\beta_j (1 \leq j \leq u - 2)$ common blocks, and three $S(2, 4, 25)$ designs with β common blocks, we have three $S(2, 4, 12u+1)$ designs with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u-2} \beta_j + \beta$ common blocks, where $\beta_j \in J_3[13]$ for $1 \leq j \leq u - 2$ and $\beta \in J_3[25]$. It is checked that for any integer $n \in I_3[v]$, n can be written as the form of $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u-2} \beta_j + \beta$, where $\alpha_i \in J_{p3}[16](1 \leq i \leq b)$, $\beta_j \in J_3[13](1 \leq j \leq u - 2)$ and $\beta \in J_3[25]$. ■

Theorem 5.3 $J_3[73] = I_3[73]$.

proof. Start from an $S(2,5,25)$ design. Delete an element from this design to obtain a 5-GDD of type 4^6. Give each element of the GDD weight 3. By Lemma 4.7, there exist three 4-GDDs of type 3^5 with α common blocks, $\alpha \in \{0,1,3,15\}$. Then apply construction 3.1, to obtain three 4-GDDs of type 12^6 with $\sum_{i=1}^{24} \alpha_i$ common blocks, where $\alpha_i \in \{0,1,3,15\}$ for $1 \leq i \leq 24$. By construction 3.2 filling in the groups by three $S(2,4,13)$ designs with $B_j (1 \leq j \leq 6)$ common blocks, $\beta_j \in J_3[13]$. we have three $S(2,4,73)$ designs with $\sum_{i=1}^{24} \alpha_i + \sum_{j=1}^{6} \beta_j$ common blocks. It is checked that for any integer $n \in I_3[73]$, n can be written as the form of $\sum_{i=1}^{24} \alpha_i + \sum_{j=1}^{6} \beta_j$. ■

For the case $v = 12u + 4$ we have the following Theorems:

Theorem 5.4 $I_3[40] \setminus \{b_{40} - 15,\ b_{40} - 14\} \subseteq J_3[40]$.

proof. we use of "$v \rightarrow 3v + 1$" rule, (See [18]). Let $(\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$ design, and let $\mathcal{V}'$ be a set such that $|\mathcal{V}'| = 2v + 1$, $\mathcal{V}' \cap \mathcal{V} = \phi$. Let $(\mathcal{V}', \mathcal{C})$ be a resolvable $STS(2v+1)$ and let $\mathcal{R} = \{R_1, \ldots, R_v\}$ be a resolution of $(\mathcal{V}', \mathcal{C})$, that is, let $(\mathcal{V}', \mathcal{C}, \mathcal{R})$ be a Kirkman triple system of order $2v + 1$; since $v \equiv\ 1,4 \pmod{12}$, such a system exists. Form the set of quadruples $D_i = \{\{v_i, x, y, z\} : v_i \in \mathrm{V}, \{x, y, z\} \in R_i\}$, and put $\mathcal{D} = \bigcup_i D_i$. Then $(\mathcal{V} \cup \mathcal{V}', \mathcal{B} \cup \mathcal{D})$ is an $S(2,4,3v+1)$ design.

Now let $v = 13$ and $(\mathcal{V}', \mathcal{C})$ be a $KTS(27)$ containing three disjoint Kirkman triple systems of order 9. Let $R_1, \ldots, R_4, R_5, \ldots, R_{13}$ are the 13 parallel classes of the $KTS(27)$ so that $R_1, \ldots, R_4$ each induce parallel classes in the three $KTS(9)$'s. We add 13 elements $a_1, \ldots, a_4, b_1, \ldots, b_9$ to this $KTS(27)$ and form blocks by adding a_i to each triple in R_i $(i = 1, \ldots, 4)$ and b_i to each triple in R_i $(i \geq 5)$. Finally, place an $S(2,4,13)$ design on the 13 new elements. Consider each ingredient in turn. on the $S(2,4,13)$ design we can get any intersection size from $J_3[13]$. On the (b_i, R_i) blocks, we can permute the R_i to obtain intersection numbers $\{0, 9, 18, 27, 36, 45, 54, 81\}$. We do not have 63 in this set because there exist three designs for intersection and we must permute at least three parallel classes. On the (a_i, R_i) blocks, we can permute the parallel classes of each of the $KTS(9)$'s to obtain intersection numbers $\{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 36\}$. It is checked that for any integer $n \in I_3[40]$, n can be written as the sum of these numbers except $b_{40} - 16$, $b_{40} - 15$, and $b_{40} - 14$.

For $b_{40} - 16$:

Start from a 4-GDD of type 3^4 from Lemma 3.6, Give each element of the GDD weight 3. By lemma 4.8 there exist three 4-GDDs of type 3^4 with α common blocks, $\alpha \in \{0,1,9\}$. Then apply construction 3.1 to obtain three 4-GDDs of type 9^4 with $\sum_{i=1}^{9} \alpha_i$ common blocks, where $\alpha_i \in \{0,1,9\}$ for $1 \leq i \leq 9$. By construction 3.3 filling in the groups by three $S(2,4,13)$ designs with β_j $(1 \leq j \leq 4)$ common blocks, $\beta_j \in J_3[13]$. We have three

$S(2,4,v)$ designs with $\sum_{i=1}^{9}\alpha_i+\sum_{j=1}^{4}\beta_j-3$ common blocks. $b_{40}-16$ can be written as this form. ■

Theorem 5.5 $J_3[76]=I_3[76]$.

proof. Using a 5-GDD of type 5^5 (the 2-(25,5,1) design itself); Give each element of the GDD weight 3. By Lemma 4.7, there exist three 4-GDDs of type 3^5 with α common blocks, $\alpha\in\{0,1,3,15\}$. Then apply construction 3.1, to obtain three 4-GDDs of type 15^5 with $\sum_{i=1}^{25}\alpha_i$ common blocks, where $\alpha_i\in\{0,1,3,15\}$ for $1\le i\le 25$. By construction 3.2, filling in the groups by three $S(2,4,16)$ designs with $B_j(1\le j\le 5)$ common blocks, we have three $S(2,4,76)$ designs with $\sum_{i=1}^{25}\alpha_i+\sum_{j=1}^{5}\beta_j$ common blocks, where $\alpha_i\in\{0,1,3,15\}$ for $1\le i\le 5$ and $\beta_j\in J_3[16]$. It is checked that for any integer $n\in I_3[76]$, n can be written as the form of $\sum_{i=1}^{25}\alpha_i+\sum_{j=1}^{5}\beta_j$, except $b_{76}-8, b_{76}-9, b_{76}-10, b_{76}-11, b_{76}-13, b_{76}-21, b_{76}-22, b_{76}-23$. Now we must handle the remaining values. Rees and Stinson (See [17]) proved that if $v\equiv 1,4 \pmod{12}$, $w\equiv 1,4 \pmod{12}$ and $v\ge 3w+1$, then there exists an $S(2,4,v)$ design contains an $S(2,4,w)$ subdesign. By taking all blocks not in the subdesign identically, and three copies of the subdesign intersecting in all but s blocks we have that $b_v-s\in J_3[v]$ if $b_w-s\in J_3[v]$. Using this result with $w=13$, we obtain intersection numbers $b_v-8, b_v-9, b_v-10, b_v-11, b_v-13$ for $v\ge 40$. Similarly using $w=25$, we obtain $b_v-21\in J_3[v]$ for $v\ge 76$.
There exists a 4-GDD of type $12^5 15^1$ from Lemma 3.8. Filling in the groups with five $S(2,4,13)$ designs and one $S(2,4,16)$ design. Hence we have an $S(2,4,76)$ design. This design has five $S(2,4,13)$ subdesigns intersecting in a single element. By choosing suitable intersection sizes from $J_3[13]$ we can obtain $\{b_{76}-22, b_{76}-23\}\subset J_3[76]$. ■

Lemma 5.6 *(i)* $\{b_v-21, b_v-22, b_v-23, b_v-25\}\subset J_3[v]$*, for* $v=52$ *and* 64*.*
(ii) $\{b_v-22, b_v-23, b_v-25\}\subset J_3[v]$*, for* $v=88, 100,$ *and* 112*.*

proof. **i**, for $v=52$, observe that there exists a GDD on 52 elements with block size 4 and group type 13^4 (See [8]). Construct three $S(2,4,52)$ designs, take the blocks of GDD identically. Replace each of the four groups by three $S(2,4,13)$ designs. By choosing suitable intersection sizes from $J_3[13]$, we get $\{b_{52}-21, b_{52}-22, b_{52}-23, b_{52}-25\}\subset J_3[52]$.
Consider $v=64$. Let G, B be a GDD on 21 elements with block size 4 and 5, and group type $5^1 4^4$ (See [8]). Apply Lemma 3.5 to produce $S(2,4,64)$ design. This design has four $S(2,4,13)$ subdesigns intersecting in a single element, and by choosing suitable intersection sizes from $J_3[13]$. We have

$\{b_{64}-21, b_{64}-22, b_{64}-23, b_{64}-25\} \subset J_3[64]$.
ii, There exists a 4-GDD of type $12^{u-1}15^1$ from Lemma 3.8, for $u = 7, 8$, and 9. Filling in the groups with $S(2,4,13)$ designs and one $S(2,4,16)$ design. Hence we have an $S(2,4,12u+4)$ design. This design has $u-1$, $S(2,4,13)$ subdesigns intersecting in a single element. By choosing suitable intersection sizes from $J_3[13]$ we can obtain $\{b_{12u+4}-22, b_{12u+4}-23, b_{12u+4}-25\} \subset J_3[12u+4]$, for $u = 7, 8$, and 9. ■

Theorem 5.7 *For any positive integer $v = 12u+4$, $u \equiv 0, 1 \pmod 4)$, u ≥ 4, $J_3[v] = I_3[v]$.* **proof.** T*here exists a 4-GDD of type 3^u with $b = \frac{3u(u-1)}{4}$ blocks. By Lemma 3.4, we have three $S(2,4,12u+4)$ designs with $\sum_{i=1}^{b}\alpha_i + \sum_{j=1}^{u}\beta_j$ common blocks, where $\beta_j + 1 \in J_3[16]$ for $1 \leq j \leq u-1$, $\beta_u \in J_3[16]$ and $\alpha_i \in J_{p3}[16]$ for $1 \leq i \leq b$. This produce all values except $b_v-8, b_v-9, b_v-10, b_v-11, b_v-13, b_v-21, b_v-22, b_v-23$, and b_v-25. By a similar argument as in Lemma 5.5 we have $\{b_v-8, b_v-9, b_v-10, b_v-11, b_v-13\} \subset J_3[v] (v \geq 40)$, and $b_v-21 \in J_3[v]$ $(v \geq 76)$.*
$b_v-21 \in J_3[v]$ for $v = 52$ and 64 by Lemma 5.6.
But b_v-22, b_v-23, and b_v-25: We know, $I_3[40] - \{b_{40}-14, b_{40}-15\} \subseteq J_3[40]$, By Ress and Stinson theorem $\{b_v-22, b_v-23, b_v-25\} \subset J_3[12u+4]$, for all u that $12u+4 \geq 3\times40+1 \Longrightarrow u \geq 10$. Hence it remains to prove that $\{b_v-22, b_v-23, b_v-25\} \subset J_3[12u+4]$ for $4 \leq u \leq 9$ $(u \equiv 0, 1 \pmod 4))$, that it is proved in Lemma 5.6. ■

Theorem 5.8 *For any positive integer $v = 12u+4$, $u \equiv 2, 3 \pmod 4)$, u ≥ 7, $J_3[v] = I_3[v]$.* **proof.** B*y proof of Theorem 5.2, there exists a 4-GDD of type $3^{u-2}6^1$. From Lemma 3.4, we have three $S(2,4,12u+4)$ designs with $\sum_{i=1}^{b}\alpha_i + \sum_{j=1}^{u-2}\beta_j + d_1$ common blocks, where $\beta_j + 1 \in J_3[16]$ for $1 \leq j \leq u-3$, $\beta_{u-2} \in J_3[16]$, $\alpha_i \in J_{p3}[16]$ for $1 \leq i \leq b$ and $d_1 + 1 \in J_3[28]$. Like the previous case we have all intersection numbers except b_v-22, b_v-23, and b_v-25: By a similar argument as in Theorem 5.7, we have $\{b_v-22, b_v-23, b_v-25\} \subset J_3[12u+4]$, for all u that $12u+4 \geq 3\times40+1 \Longrightarrow u \geq 10$. Hence it remains to prove that $\{b_v-22, b_v-23, b_v-25\} \subset J_3[12u+4]$ for $7 \leq u \leq 9$ $(u \equiv 2, 3 \pmod 4))$, that it is proved in Lemma 5.6.* ■

6 Small Orders

Three small orders, $\{25, 28, 37\}$, remain. We use some techniques to determine situation of half of numbers that those can be as intersection numbers.

In the next example we discuss a method which may help in understanding a general method in the following theorems.

Example 6.1 Construct an $S(2,4,25)$ design, $(\mathcal{V},\mathcal{B})$ with $\mathcal{V}=\mathcal{Z}_{25}$. In this $S(2,4,25)$ design, the elements $\{1,2,3,5,6,8,9\}$ induce an $STS(7)$. All blocks of $\mathcal{B}$ are listed in the following, which can be found in [19] (design 17).
We divide these blocks to three parts A, B and C. A contains the blocks that induce the $STS(7)$. B contains the blocks that do not contain any element of the $STS(7)$ and C contains the remanded blocks.

A :

$1,2,3,4$	$1,5,6,7$	$1,8,9,10$	$2,5,8,11$
$2,6,9,14$	$3,5,9,24$	$3,6,8,22.$	

B :

$4,10,16,25$	$4,11,17,21$	$4,15,22,24$	$7,10,19,24$
$7,12,14,18$	$10,13,14,22$	$11,14,20,24$	$7,11,22,23.$

C :

$1,11,12,13$	$1,14,15,16$	$1,17,18,19$	$1,20,21,22$	$1,23,24,25$
$2,7,15,17$	$2,10,12,20$	$2,13,16,23$	$2,18,21,24$	$2,19,22,25$
$3,7,13,25$	$3,10,11,18$	$3,12,15,19$	$3,14,21,23$	$3,16,17,20$
$5,14,4,19$	$5,10,17,23$	$5,12,21,25$	$5,13,15,20$	$5,16,18,22$
$9,4,13,18$	$9,7,16,21$	$9,11,15,25$	$9,12,17,22$	$9,19,20,23$
$8,4,7,20$	$8,12,16,24$	$8,13,19,21$	$8,14,17,25$	$8,15,18,23$
$6,4,12,23$	$6,10,15,21$	$6,11,16,19$	$6,13,17,24$	$6,18,20,25.$

Consider the permutation $\pi=(1,2,3)(18,17,16,13,12,11)$. This permutation consists of two parts the first part $\pi_1=(1,2,3)$ contains some elements of the $STS(7)$ and the second part $\pi_2=(18,17,16,13,12,11)$ does not contain any element of the $STS(7)$. When π and π^{-1} act on A we have 1 as intersection number on A, $\pi(A)$, and $\pi^{-1}(A)$.

A	$\pi(A)$	$\pi^{-1}(A)$
1,2,3,4	**1,2,3,4**	**1,2,3,4**
$1,5,6,7$	$2,5,6,7$	$3,5,6,7$
$1,8,9,10$	$2,8,9,10$	$3,8,9,10$
$2,5,8,11$	$3,5,8,18$	$1,5,8,12$
$2,6,9,14$	$3,6,9,14$	$1,6,9,14$
$3,5,9,24$	$1,5,9,24$	$2,5,9,24$
$3,6,8,22$	$1,6,8,22$	$2,6,8,22$

But when π and π^{-1} act on $\mathcal{B}\setminus A$, we have 6 as intersection number on $\mathcal{B}\setminus A$, $\pi(\mathcal{B}\setminus A)$, and $\pi^{-1}(\mathcal{B}\setminus A)$. Then we get intersection number $7=1+6$ on $\mathcal{B}$, $\pi(\mathcal{B})$ and $\pi^{-1}(\mathcal{B})$.

$\mathcal{B} \setminus A$:

1, 11, 12, 13	1, 14, 15, 16	1, 17, 18, 19	1, 20, 21, 22	1, 23, 24, 25
2, 7, 15, 17	2, 10, 12, 20	2, 13, 16, 23	2, 18, 21, 24	2, 19, 22, 25
3, 7, 13, 25	3, 10, 11, 18	3, 12, 15, 19	3, 14, 21, 23	3, 16, 17, 20
5,14,4,19	5, 10, 17, 23	5, 12, 21, 25	5, 13, 15, 20	5, 16, 18, 22
9, 4, 13, 18	9, 7, 16, 21	9, 11, 15, 25	9, 12, 17, 22	**9,19,20,23**
8,4,7,20	8, 12, 16, 24	8, 13, 19, 21	8, 14, 17, 25	8, 15, 18, 23
6, 4, 12, 23	**6,10,15,21**	6, 11, 16, 19	6, 13, 17, 24	6, 18, 20, 25.
4, 10, 16, 25	4, 11, 17, 21	**4,15,22,24**	**7,10,19,24**	7, 11, 22, 23.
7, 12, 14, 18	10, 13, 14, 22	11, 14, 20, 24		.

$\pi(\mathcal{B} \setminus A)$:

$\underline{2}$, 18, 11, 12	$\underline{2}$, 14, 15, 13	$\underline{2}$, 16, 17, 19	$\underline{2}$, 20, 21, 22	$\underline{2}$, 23, 24, 25
$\underline{3}$, 7, 15, 16	$\underline{3}$, 10, 11, 20	$\underline{3}$, 12, 13, 23	$\underline{3}$, 17, 21, 24	$\underline{3}$, 19, 22, 25
$\underline{1}$, 7, 12, 25	$\underline{1}$, 10, 18, 17	$\underline{1}$, 11, 15, 19	$\underline{1}$, 14, 21, 23	$\underline{1}$, 13, 16, 20
6, 4, 11, 23	**6,10,15,21**	6, 18, 13, 19	6, 12, 16, 24	6, 17, 20, 25
5,14,4,19	5, 10, 16, 23	5, 11, 21, 25	5, 12, 15, 20	5, 13, 17, 22
8, 11, 13, 24	8, 12, 19, 21	8, 14, 16, 25	8, 15, 17, 23	**8,4,7,20**
9, 4, 12, 17	9, 7, 13, 21	9, 18, 15, 25	9, 11, 16, 22	**9,19,20,23**
4, 10, 13, 25	4, 18, 16, 21	**4,15,22,24**	**7,10,19,24**	7, 18, 22, 23
7, 11, 14, 17	10, 12, 14, 22	18, 14, 20, 24		.

$\pi^{-1}(\mathcal{B} \setminus A)$:

$\underline{3}$, 12, 13, 16	$\underline{3}$, 14, 15, 17	$\underline{3}$, 18, 11, 19	$\underline{3}$, 20, 21, 22	$\underline{3}$, 23, 24, 25
$\underline{1}$, 7, 15, 18	$\underline{1}$, 10, 13, 20	$\underline{1}$, 16, 17, 23	$\underline{1}$, 11, 21, 24	$\underline{1}$, 19, 22, 25
$\underline{2}$, 7, 16, 25	$\underline{2}$, 10, 12, 11	$\underline{2}$, 13, 15, 19	$\underline{2}$, 14, 21, 23	$\underline{2}$, 17, 18, 20
6, 4, 13, 23	**6,10,15,21**	6, 12, 17, 19	6, 16, 18, 24	6, 11, 20, 25
5,14,4,19	5, 10, 18, 23	5, 13, 21, 25	5, 16, 15, 20	5, 17, 18, 22
8, 13, 17, 24	8, 16, 19, 21	8, 14, 18, 25	8, 15, 11, 23	**8,4,7,20**
9, 4, 16, 11	9, 7, 17, 21	9, 12, 15, 25	9, 13, 18, 22	**9,19,20,23**
4, 10, 17, 25	4, 12, 18, 21	**4,15,22,24**	**7,10,19,24**	7, 12, 22, 23
7, 13, 14, 11	10, 16, 14, 22	12, 14, 20, 24		.

In fact we obtain two intersection numbers, the first number is obtained when π_1 and π_1^{-1} act on A. The second number is obtained when π_2 and π_2^{-1} act on $\mathcal{B} \setminus A$. Then we add these numbers and obtain intersection number of $\mathcal{B}$, $\pi(\mathcal{B})$, and $\pi^{-1}(\mathcal{B})$. Since the common blocks of A, $\pi_1(A)$, and $\pi_1^{-1}(A)$ do not contain any element of π_2 and common blocks of $\mathcal{B} \setminus A$, $\pi_2(\mathcal{B} \setminus A)$, and $\pi_2^{-1}(\mathcal{B} \setminus A)$ do not contain any element of π.
Note, we choose some permutations which change at most two elements of each block. Also the design is Steiner, so when the block b changes, it is commuted to different block from the other blocks. Hence by applying permutations, no new common block form in $\pi(\mathcal{B})$ and $\pi^{-1}(\mathcal{B})$. (We separate

B and C for choosing suitable permutations.)

Lemma 6.2 $[0,11] \cup \{13,15,17,20,29,50\} \cup [22,24] \subseteq J_3[25]$ *and* $\{42\} \notin J_3[25]$.

proof. Take the design $(\mathcal{V},\mathcal{B})$ which is stated in Example 6.1. We get these intersection numbers $[5,8] \cup \{11,13,15,17,20,24,29\}$ by the method of Example 6.1 on $(\mathcal{V},\mathcal{B})$. Also we get these intersection numbers: $[0,10] \cup \{22,23,29\}$, with applying straight permutations on $\mathcal{B}$.
We have $42 \notin J_3[25]$ since $42 \notin J_2[25]$. This completes proof. ■

Lemma 6.3 $[1,24] \cup \{27,28,33,37,39,63\} \subseteq J_3[28]$.

proof. We obtain these intersection numbers in two steps.
Step 1:
Construct an $S(2,4,28)$ design, $(\mathcal{V},\mathcal{B})$ with $\mathcal{V} = \mathcal{Z}_{28}$. All blocks of $\mathcal{B}$ are listed in the following, which can be found in Theorem 20 in [14]. In this $S(2,4,28)$ design the elements $\{2,4,16,22,25,26,27\}$ induce an $STS(7)$. By a similar argument in Example 6.1, we obtain these intersection numbers: $[2,7] \cup [10,12] \cup [16,19] \cup [21,24] \cup \{14,27,28,33,37,39\}$.

2, 0, 1, 3	4, 5, 6, 7	16, 17, 18, 19	22, 20, 21, 23	25, 7, 8, 1
2, 7, 18, 23	4, 11, 14, 23	16, 3, 11, 13	22, 3, 6, 19	25, 5, 14, 18
2, 10, 12, 17	4, 0, 8, 12	16, 0, 5, 20	22, 5, 10, 15	25, 9, 19, 20
2, 13, 20, 24	4, 1, 17, 21	16, 1, 6, 23	22, 0, 7, 17	25, 0, 15, 23
2, 5, 19, 21	4, 15, 19, 24	16, 9, 15, 21	22, 9, 12, 18	25, 6, 13, 17
2, 8, 6, 15	4, 3, 18, 20	16, 7, 12, 24	22, 1, 14, 24	25, 3, 12, 21
26, 1, 15, 18	27, 3, 15, 17	8, 9, 10, 11	0, 6, 9, 24	24, 25, 26, 27
26, 7, 10, 19	27, 5, 12, 23	12, 13, 14, 15	0, 10, 13, 18	9, 2, 4, 27
26, 9, 17, 23	27, 1, 10, 20	3, 7, 9, 14	1, 11, 12, 19	11, 2, 22, 25
26, 3, 8, 5	27, 0, 14, 19	3, 10, 23, 24	6, 10, 14, 21	14, 2, 16, 26
26, 12, 6, 20	27, 11, 6, 18	5, 11, 17, 24	7, 11, 15, 20	10, 4, 16, 25
26, 0, 11, 21	27, 7, 13, 21	1, 5, 9, 13	8, 14, 17, 20	13, 4, 22, 26
8, 16, 22, 27	8, 13, 19, 23	8, 18, 21, 24.		

Step 2:
Take an $S(2,4,28)$ design, $(\mathcal{V},\mathcal{B})$ with $\mathcal{V} = \mathcal{Z}_{28}$. In this $S(2,4,28)$ design, the elements $\{4,5,6,13,14,15,19,20,21\}$ induce an $STS(9)$. All blocks of $\mathcal{B}$ are listed in the following, which can be found in Theorem 21 in [14].

$4,0,1,7$	$5,0,2,8$	$6,10,12,25$	$13,0,10,16$	$14,12,16,22$
$4,2,3,12$	$5,1,3,10$	$6,8,22,23$	$13,1,8,9$	$14,8,10,24$
$4,8,17,25$	$5,9,18,26$	$6,7,16,27$	$13,2,17,27$	$14,3,18,25$
$4,9,23,24$	$5,7,22,24$	$6,1,2,11$	$13,7,12,23$	$14,2,7,9$
$4,10,11,26$	$5,11,12,27$	$6,0,3,9$	$13,11,18,24$	$14,0,11,17$
$15,0,12,18$	$19,0,22,25$	$20,8,18,27$	$21,0,27,24$	$1,17,18,22$
$15,3,7,8$	$19,1,12,24$	$20,7,11,25$	$21,3,11,23$	$1,23,25,27$
$15,1,16,26$	$19,8,11,16$	$20,9,12,17$	$21,7,10,18$	$2,24,25,26$
$15,9,11,22$	$19,9,10,27$	$20,2,10,22$	$21,8,12,26$	$3,22,26,27$
$15,10,17,23$	$19,7,17,26$	$20,0,23,26$	$21,9,16,25$	$3,17,16,24$
$1,14,20,21$	$3,13,19,20$	$4,5,16,20$	$5,13,15,25$	$4,13,21,22$
$2,15,19,21$	$4,6,18,19$	$4,14,15,27$	$6,13,14,26$	$5,14,19,23$
$2,18,23,16$	$5,6,17,21$	$6,15,20,24.$		

By previous method we obtain these intersection numbers $[5,17] \cup [19,21] \cup \{23,24,28,33,39\}$, in this step.
Also we obtain 1 as intersection number in Lemma 4.4. ■

Lemma 6.4 $\{18,19,78,79,81,87,102,103,111\} \cup [21,32] \cup [34,36] \cup [38,43] \cup [45,48] \cup [52,54] \cup [58,63] \cup [67,71] \subseteq J_3[37]$

proof. In this Lemma we have three steps.
Step 1:
Take an $S(2,4,37)$ design, $(\mathcal{V},\mathcal{B})$ with
$\mathcal{V} = \{a_0,\cdots,a_8,b_0,\cdots,b_8,c_0,\cdots,c_8,d_0,\cdots,d_8,\infty\}$. Develop the following base blocks over $\mathcal{Z}_9$ to obtain all blocks of $\mathcal{B}$ (See [12]). In this $S(2,4,37)$ design the elements $\{a_0,a_3,a_6,b_0,b_3,b_6,c_0,c_3,c_6\}$ induce an $STS(9)$.

$\{\infty,a_0,a_3,a_6\}$, $\{\infty,b_0,b_3,b_6\}$, $\{\infty,c_0,c_3,c_6\}$, $\{\infty,d_0,d_3,d_6\}$
$\{a_0,a_1,b_3,c_0\}$, $\{a_0,a_5,b_6,c_6\}$, $\{a_0,a_7,d_0,d_1\}$, $\{a_0,b_0,b_4,c_3\}$
$\{a_1,c_3,c_8,d_0\}$, $\{a_2,c_6,c_7,d_0\}$, $\{a_3,b_8,d_0,d_7\}$, $\{a_4,b_2,b_3,d_0\}$
$\{b_0,c_1,d_0,d_5\}$, $\{b_5,b_7,c_0,d_0\}$, $\{b_6,c_2,c_4,d_0\}$.

By a similar argument in Example 6.1, we obtain these intersection numbers:
$\{18,19,21,22,69,70,78,81\} \cup [24,32] \cup [34,36] \cup [38,43] \cup [45,48] \cup [52,54] \cup [60,62]$.
Step 2:
Construct an $S(2,4,37)$ design, $(\mathcal{V},\mathcal{B})$ with
$\mathcal{V} = \mathcal{Z}_{11} \times \{1,2,3\} \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$. In this $S(2,4,37)$ design the elements $\{0_1,1_1,2_2,10_2,3_3,4_3,5_3\}$ induce an $STS(7)$. Develop the following base blocks over $\mathcal{Z}_{11}$ to obtain all blocks of $\mathcal{B}$ ($\infty_1,\infty_2,\infty_3$ and ∞_4 are constants) (See [14]). By a similar argument in Example 6.1, we obtain these intersection numbers:

$\{23, 26, 29, 32, 35, 36, 38, 39, 42, 43, 45, 47, 48, 53, 54, 60, 61, 68, 69, 78\}$.

$\{0_1, 0_2, 0_3, \infty_1\}$, $\{0_1, 1_2, 2_3, \infty_2\}$, $\{0_1, 2_2, 5_3, \infty_3\}$
$\{0_1, 8_2, 6_3, \infty_4\}$, $\{0_1, 1_1, 5_1, 10_2\}$, $\{0_2, 2_2, 5_2, 7_3\}$
$\{8_1, 0_3, 1_3, 5_3\}$, $\{0_1, 3_1, 6_2, 7_2\}$, $\{0_2, 4_2, 8_3, 10_3\}$.
$\{2_1, 4_1, 0_3, 3_3\}$
Also the design contains the block $\{\infty_1, \infty_2, \infty_3, \infty_4\}$.

Step 3: Take an $S(2, 4, 37)$ design, $(\mathcal{V}, \mathcal{B})$ with $\mathcal{V} = \{\infty\} \cup (\{x, y, z\} \times \mathcal{Z}_{12})$. By developing the following base blocks over $\mathcal{Z}_{12}$ we get the main part of the blocks (See [8]):

$\{z_0, x_0, y_0, \infty\}$, $\{x_0, x_4, y_{11}, z_5\}$, $\{x_2, z_0, z_1, z_5\}$
$\{x_7, y_0, y_1, z_9\}$, $\{x_{10}, y_0, y_2, z_4\}$, $\{x_3, y_0, y_4, z_7\}$
$\{x_2, y_0, y_5, z_{10}\}$, $\{x_5, y_1, z_0, z_2\}$.
and the short orbits:
$\{y_0, y_3, y_6, y_9\}$, $\{z_0, z_3, z_6, z_9\}$.

Call the resulting set of 102 blocks B and call the other set of blocks C. C contains nine blocks which covers the remaining pairs. In fact C comes from $S(2, 4, 13)$ design with omitting one flower. This enable us to replace C by a different set C' or C'' of blocks covering the same pairs, So in this part we can have intersection number $C \cap C' \cap C''$. Recall that $C \cap C' \cap C''$ can be any of $\{0, 1, 9\} \subseteq J_{f3}[13]$. Also we consider some permutations on B which be used in [8] and those are suitable for three designs. Let π be one of them. We construct $B' = \pi(B)$ and $B'' = \pi^{-1}(B)$. Hence we obtain intersection sizes $|B \cap B' \cap B''| + i,\ i \in \{0, 1, 9\}$. Now we get in this step these intersection numbers $\{58, 59, 62, 63, 67, 78, 79, 87, 102, 103, 111\} \cup [69, 71]$. ■

7 conclusion

In this paper, we have obtained the complete solution of the intersection problem for three $S(2, 4, v)$ designs with $v = 13, 16$ and $v \geq 49$.
Proof of Theorem 1.1:
(1): By Lemma 2.1 we have $J_3[v] \subseteq I_3[v]$.
(2): By combining the results of Theorems 5.1, 5.2, 5.3, 5.5, 5.7, and 5.8 we have $J_3[v] = I_3[v]$ for all admissible $v \geq 49$.
(3): By Theorem 5.4, we obtain $I_3[40] \setminus \{b_{40} - 15, b_{40} - 14\} \subseteq J_3[40]$.
(4): It holds by Lemmas 4.1 and 4.2.
(5), (6), and (7): We prove these sentences in the last section.

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